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SPECTRAL ANALYSIS Of THE DISCONTINUOUS STURM-LIOUVILLE OPERATOR WITH ALMOST-PERIODIC POTENTIALS

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Abstract The spectral analysis for the Schrodinger operator with complex almost periodic potentials and the discontinuous coefficient on the axis are studied. Investigated the main characteristics of the fundamental solutions and the spectrum of the operator is analyzed. The inverse problem is formulated, a uniqueness theorem is proved, a constructive procedure for the solution of the inverse problem is provided.

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1 Introduction

The purpose of the presented work is the spectral analysis of the differential operator L which is generated by the expression

$$l(y) \equiv \frac{1}{\rho(x)}[-\frac{d^2}{dx^2} + q(x)] \tag{1}$$

in the space $L_2(R)$, where

$$\rho(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}, \tag{2}$$

$$q(x) = \sum_{n=1}^{\infty} q_n e^{i\Lambda_n x} \tag{3}$$

and the condition

$$\sum_{n=1}^{\infty} |q_n| < \infty \tag{4}$$

is satisfied.

The set of exponents is a countable set of positive real numbers closed to the addition

$$M = \{\Lambda_1, \Lambda_2, \Lambda_3, ...\Lambda_n, ...\}, \quad \Lambda_n > 0, \; n \in N. \tag{5}$$

Let $A(M)$ be a Banach algebra of almost periodic functions

$$A(M) = \left\{ f:\ f(t) = \sum_{g\in M} f_g e^{igt} \right\},\quad \|f\| = \sum_{g\in M} |f_g| \tag{6}$$

We define a new class $Q(M) = \left\{ q:\ q(x) = \sum_{n\in N} q_n e^{i\Lambda_n x},\ \sum_{n\in N} |q_n| \Lambda_n^{-1} < \infty \right\}$ of Besicovitch almost periodic functions such that the second primitive of (3) exists and belongs to $A(M)$.

Note that algebra $A(M)$ will play an important role in our investigation and the equality (6) will be called the basic condition.

We write $\alpha \gg \beta$ or $\alpha \ll \beta$ if $\Lambda_\alpha > \Lambda_\beta$ or $\Lambda_\alpha < \Lambda_\beta$ respectively and the symbol $\sum_{\alpha:\alpha>\beta}$ means summing over all α such that $\Lambda_\alpha > \Lambda_\beta$, also we will use $\alpha \oplus \beta = \gamma$ if $\Lambda_\alpha + \Lambda_\beta = \Lambda_\gamma$

For the Sturm–Liouville operator with different singularities (i.e. on the half-axis having an arbitrary number of turning points, having singularities and turning points at the end-points of the interval) the determination of the spectral function or normalizing constants has been studied by Freiling & Yurko (2001). These results are mainly based on Volterra operator transformation and contour integrations.

The string equation or wave equation in layered medium was investigated by Krein (1955), Blagoveshchensky (1969), Grinberg (1990), Mamedov (2021), Demirbilek & Mamedov (2021) and others.

Belishev (1987) first considered (1) with $q(x) = 0$ and solved the inverse problem of reconstructing the $\rho(x)$ of a finite-inhomogeneous string from the frequencies and energies of its normalized characteristic vibrations in the case where $\rho(x)$ can change its sign (is indefinite).

For periodic potentials (3) (i.e. $\Lambda_n = n$) this problem was studied by Efendiev (2011) where the main characteristics of the fundamental solutions are investigated, the spectrum of the operator is analyzed.

The interest in the investigation of the spectral properties of the differential operators with periodic coefficients has been increased after the study of Gasymov (1980).

In the cited paper the special solutions of the equation $-y'' + q(x)y = \lambda^2 y$ were constructed and the spectral data $\{s_n\}_{n\in N}$ were determined. It was shown that the inverse problem reconstructing of the operator L from this sequence has a solution. Afterwards, the result of Gasymov (1980) was extended by Gasymov & Orudzhev (1986) to almost-periodic potentials of the form (3). The operator L generated by the finite sum in (3) was studied by Sarnak (1982) who, in particular, showed that the spectrum L always coincides with $[0,\infty)$.

2 Properties of the Floquet solution of the equation $l(y) = \lambda^2 y$

The existence of the Floquet solutions of the equation $l(y) = \lambda^2 y$ which will play an important role in the investigation of the spectrum of the operator L was considered in Bahlulzadeh (2017), where it was shown that $f_1^{\pm}(x,\lambda)$ and $f_2^{\pm}(x,\lambda)$ are the solutions of the equation

$$-y'' + q(x)y = \lambda^2 \rho(x)\, y,\ \ \lambda \text{ - is a complex parameter} \tag{7}$$

satisfy the following conditions

$$\lim_{Imx\to+\infty} f_1^{\pm}(x,\lambda)\, e^{\mp i\lambda x} = 1 \qquad for \quad \pm Im\lambda > 0$$

$$\lim_{Imx\to-\infty} f_2^{\pm}(x,\lambda)\, e^{\mp \lambda x} = 1 \qquad for \quad \pm Re\lambda > 0,$$

and fulfill the following theorem

Theorem 1. *Equation (7) with potential $q(x) \in Q(M)$ and $\rho(x)$ defined as (2) has the particular solutions of the form*

$$f_1^{\pm}(x,\lambda) = e^{\pm i\lambda x}\left(1+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{\Lambda_n \pm 2\lambda}e^{i\Lambda_\alpha x}\right) \quad for \;\; x \geq 0 \tag{8}$$

$$f_2^{\pm}(x,\lambda) = e^{\pm \lambda x}\left(1+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{\Lambda_n \mp 2i\lambda}e^{i\Lambda_\alpha x}\right) \quad for \;\; x < 0 \tag{9}$$

here the numbers $V_{n\alpha}$ are determined from the following relations

$$\Lambda_\alpha\left(\Lambda_\alpha - \Lambda_n\right)V_{n\alpha} + \sum_{\beta\oplus\gamma=n} V_{n\beta}q_\gamma = 0 \tag{10}$$

$$q_\alpha + \sum_{\beta\oplus\gamma=n} V_{n\beta}q_\gamma = 0. \tag{11}$$

and series

$$\sum_{n=1}^{\infty}\Lambda_n^{-1}\sum_{\alpha=n}^{\infty}\Lambda_\alpha\left(\Lambda_\alpha - \Lambda_n\right)|V_{n\alpha}| \tag{12}$$

converges.

We easily see that at the points $\lambda = \mp\frac{\Lambda_n}{2}, \left(\lambda = \pm\frac{i\Lambda_n}{2}\right)$ $n \in N$ there can be simple poles to the function $f_1^{\pm}(x,\lambda)(f_2^{\pm}(x,\lambda))$

Remark 1. *If $\lambda \neq -\frac{\Lambda_n}{2}$ and $Im\lambda > 0$, then $f_1^{+}(x,\lambda) \in L_2(0,\infty)$.*

Remark 2. *If $\lambda \neq -\frac{i\Lambda_n}{2}$ and $Re\lambda > 0$, then $f_2^{+}(x,\lambda) \in L_2(-\infty,0)$.*

Taking into account that the potential $q(x)$ can be extended to the upper semi-plane as an analytic function, we find

$$W[f_1^{+}(x,\lambda), f_1^{-}(x,\lambda)] = -2i\lambda \neq 0, \;\; for \quad \lambda \neq 0, \pm\frac{\Lambda_n}{2} \tag{13}$$

$$W[f_2^{+}(x,\lambda), f_2^{-}(x,\lambda)] = -2\lambda \neq 0, \;\; for \quad \lambda \neq 0, \pm\frac{i\Lambda_n}{2}. \tag{14}$$

Therefore, the functions $f_1^{+}(x,\lambda), f_1^{-}(x,\lambda)$ $(f_2^{+}(x,\lambda), f_2^{-}(x,\lambda))$ are linearly independent solutions of the equation (7) for $\lambda \neq 0, \pm\frac{\Lambda_n}{2}, (\pm\frac{i\Lambda_n}{2})$.

Consequently, any solution of the equation (7) corresponding to $x \geq 0$ $(x < 0)$ can be represented as a linear combination of the solutions $f_1^{+}(x,\lambda), f_1^{-}(x,\lambda)$, $(f_2^{+}(x,\lambda), f_2^{-}(x,\lambda))$.

We have

$$\begin{cases} f_2^{\pm}(x,\lambda) = A^{\pm}(\lambda) f_1^{+}(x,\lambda) + B^{\pm}(\lambda) f_1^{-}(x,\lambda), \;\; x > 0 \\ f_1^{\pm}(x,\lambda) = C^{\pm}(\lambda) f_2^{+}(x,\lambda) + D^{\pm}(\lambda) f_2^{-}(x,\lambda), \;\; x < 0 \end{cases} \tag{15}$$

It means that the solutions (8) and (9) can be predetermined as

$$f_1^{\pm}(x,\lambda) = \begin{cases} e^{\pm i\lambda x}\left(1+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{\Lambda_n \mp 2\lambda}e^{i\Lambda_\alpha x}\right), & x \geq 0 \\ C^{\pm}(\lambda) f_2^{+}(x,\lambda) + D^{\pm}(\lambda) f_2^{-}(x,\lambda), & x < 0 \end{cases} \tag{16}$$

and

$$f_2^{\pm}(x,\lambda) = \begin{cases} e^{\pm \lambda x}\left(1+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{\Lambda_n \pm 2i\lambda}e^{i\Lambda_\alpha x}\right), & x < 0 \\ A^{\pm}(\lambda) f_1^{+}(x,\lambda) + B^{\pm}(\lambda) f_1^{-}(x,\lambda), & x \geq 0 \end{cases}$$

By taking into account the formulas (15) and (16) we find

$$\begin{cases} A(\lambda) = A^{+}(\lambda) = iD^{-}(\lambda) = \frac{W[f_2^{+}(x,\lambda), f_1^{-}(x,\lambda)]}{2i\lambda} & for\ \ \lambda \in S_4 \\ B(\lambda) = A^{-}(\lambda) = iC^{-}(\lambda) = \frac{W[f_2^{-}(x,\lambda), f_1^{-}(x,\lambda)]}{2i\lambda} & for\ \ \lambda \in S_3 \\ C(\lambda) = B^{+}(\lambda) = iD^{+}(\lambda) = \frac{W[f_1^{+}(x,\lambda), f_2^{+}(x,\lambda)]}{2i\lambda} & for\ \ \lambda \in S_1 \\ D(\lambda) = B^{-}(\lambda) = -iC^{+}(\lambda) = \frac{W[f_1^{+}(x,\lambda), f_2^{-}(x,\lambda)]}{2i\lambda} & for\ \ \lambda \in S_2 \end{cases} \tag{17}$$

where $S_k = \left\{ \frac{k\pi}{2} < \arg\lambda < \frac{(k+1)\pi}{2} \right\}, \ \ k = \overline{0,3}$.

Then (16) takes the following form

$$\begin{cases} f_2^{+}(x,\lambda) = A(\lambda) f_1^{+}(x,\lambda) + C(\lambda) f_1^{-}(x,\lambda) \\ f_2^{-}(x,\lambda) = B(\lambda) f_1^{+}(x,\lambda) + D(\lambda) f_1^{-}(x,\lambda) \\ f_1^{+}(x,\lambda) = iD(\lambda) f_2^{+}(x,\lambda) - iC(\lambda) f_2^{-}(x,\lambda) \\ f_1^{-}(x,\lambda) = -iB(\lambda) f_2^{+}(x,\lambda) + iA(\lambda) f_2^{-}(x,\lambda) \end{cases} \tag{18}$$

By dividing both sides of (18) by $C(\lambda)$ and $B(\lambda)$ we obtain the solutions of the equation (7)

$$\begin{cases} U_2^{+}(x,\lambda) = \frac{A(\lambda)}{C(\lambda)} f_1^{+}(x,\lambda) + f_1^{-}(x,\lambda) \\ U_2^{-}(x,\lambda) = \frac{D(\lambda)}{B(\lambda)} f_1^{-}(x,\lambda) - f_1^{+}(x,\lambda) \\ U_1^{+}(x,\lambda) = \frac{iD(\lambda)}{C(\lambda)} f_2^{+}(x,\lambda) - i f_2^{-}(x,\lambda) \\ U_1^{-}(x,\lambda) = \frac{iA(\lambda)}{B(\lambda)} f_2^{-}(x,\lambda) - i f_2^{+}(x,\lambda) \end{cases} \tag{19}$$

According to a physical sense of the solutions, we will call $\frac{1}{B(\lambda)}$ and $\frac{1}{C(\lambda)}$ as a transmission coefficient and $\frac{A(\lambda)}{C(\lambda)}, \frac{A(\lambda)}{B(\lambda)}, \frac{D(\lambda)}{C(\lambda)}$ and $\frac{D(\lambda)}{B(\lambda)}$ as a reflection coefficient from the right and left to (7) respectively.

3 The spectrum of the operator L

To study the spectrum of the operator L at first, we calculate the kernel of the resolvent of the operator $\left(R - \lambda^2 I\right)$. First, let us prove the following theorem, from which we obtain the existence of the resolvent operator R_λ.

Theorem 2. *The operator L has no pure real and pure imaginary eigenvalues.*

Proof. The equation (7) has the fundamental solutions $f_1^{+}(x,\lambda), f_1^{-}(x,\lambda)$ $\left(f_2^{+}(x,\lambda), f_2^{-}(x,\lambda)\right)$ on $|Im\lambda| < \frac{\varepsilon}{2}$ $(|Re\lambda| < \frac{\varepsilon}{2})$ and $\lambda \neq 0, \ \ \lambda \neq \pm\frac{\Lambda_n}{2}, \ \ \lambda \neq \pm\frac{i\Lambda_n}{2}$ $, n \in N$. Then when $\lambda^2 > 0$ and $Im\lambda = 0$ the solution of the equation (7) can be written in the form of

$$y(x,\lambda) = C_1 e^{iRe\lambda x}\left(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{\Lambda_n + 2|\lambda|}e^{i\Lambda_\alpha x}\right) +$$
$$+C_2 e^{-iRe\lambda x}\left(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{\Lambda_n - 2|\lambda|}e^{i\Lambda_\alpha x}\right)$$

and when $Re\lambda = 0$

$$y(x,\lambda) = \tilde{C}_1 e^{iIm\lambda x}\left(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{\Lambda_n + 2i|\lambda|}e^{i\Lambda_\alpha x}\right) +$$
$$+\tilde{C}_2 e^{-iIm\lambda x}\left(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{\Lambda_n - 2i|\lambda|}e^{i\Lambda_\alpha x}\right)$$

In both cases, $y(x,\lambda) \notin L_2(-\infty,\infty)$, because the principal parts of the solutions are periodic. So, we proved that the operator L has no pure real and pure imaginary eigenvalues.

Note that similar results easily can be obtained for the cases when $\lambda = 0$, $\lambda = \pm\frac{\Lambda_n}{2}$, $\lambda = \pm\frac{i\Lambda_n}{2}$, $n \in N$.

We proved that, for any complex number, λ outside of $\{Re\lambda = 0\} \cup \{Im\lambda = 0\}$ there exists one to one resolvent operator $R_\lambda = \left(L - \lambda^2 I\right)^{-1}$ and using the general methods for the kernel of the resolvent, we get

$$R_{11}(x,t,\lambda) = -\frac{1}{2i\lambda C\left(\lambda\right)} \left\{ \begin{array}{ll} f_1^+(x,\lambda), f_2^+(t,\lambda), & t \le x \\ f_1^+(t,\lambda), f_2^+(x,\lambda), & t > x \end{array} \right. \quad \lambda \in S_0 \tag{20}$$

$$R_{12}(x,t,\lambda) = -\frac{1}{2i\lambda D\left(\lambda\right)} \left\{ \begin{array}{ll} f_1^+(x,\lambda), f_2^-(t,\lambda), & t \le x \\ f_1^+(t,\lambda), f_2^-(x,\lambda), & t > x \end{array} \right. \quad \lambda \in S_1 \tag{21}$$

$$R_{13}(x,t,\lambda) = \frac{1}{2i\lambda B\left(\lambda\right)} \left\{ \begin{array}{ll} f_1^-(x,\lambda), f_2^-(t,\lambda), & t \le x \\ f_1^-(t,\lambda), f_2^-(x,\lambda), & t > x \end{array} \right. \quad \lambda \in S_2 \tag{22}$$

$$R_{14}(x,t,\lambda) = \frac{1}{2i\lambda A\left(\lambda\right)} \left\{ \begin{array}{ll} f_1^-(x,\lambda), f_2^+(t,\lambda), & t \le x \\ f_1^-(t,\lambda), f_2^+(x,\lambda), & t > x \end{array} \right. \quad \lambda \in S_3 \tag{23}$$

where $S_k = \left\{\frac{k\pi}{2} < \arg\lambda < \frac{(k+1)\pi}{2}\right\}, \ \ k = \overline{0,3}$.

From the formulae (20) – (23) it is easy to see that the points $\lambda = 0$, $\lambda = \pm\frac{\Lambda_n}{2}$, $\lambda = \pm\frac{i\Lambda_n}{2}$, $n \in N$ are the poles of the resolvent, therefore, they must be eigenvalues of the operator L. According to the theorem, they don't exist. Then According to Naimark (1967), the points $\lambda^2 = \pm(\frac{\Lambda_n}{2})^2$ are spectral singularities of the operator L. □

Lemma 1. *The eigenvalues of the operator L are finite and coincide with the square of zeros of the functions $A\left(\lambda\right), B\left(\lambda\right), C\left(\lambda\right)$ and $D\left(\lambda\right)$ from the sectors S_k, $k = 0,1,2,3$ respectively.*

Proof. For the solutions, $f_1^+(0,\lambda), f_2^+(0,\lambda)$ we can obtain the asymptotic equalities

$$f_1^{\pm(j)}(0,\lambda) = \pm\left(i\lambda\right)^j C_1 + o\left(1\right), \ \ for\left|\lambda\right| \to \infty, \ \ j = 0,1, \ \ C_1 > 0$$

$$f_2^{\pm(j)}(0,\lambda) = \pm\left(\lambda\right)^j C_2 + o\left(1\right), \ \ for\left|\lambda\right| \to \infty, \ \ j = 0,1, \ \ C_2 > 0$$

For simplicity, we prove the first equality.

$$\begin{aligned}\left|f_1^{\pm(j)}(0,\lambda)\right| &= \pm\left(i\lambda\right)^j + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{\left(i\lambda\right)^j\left|V_{n\alpha}\right|}{\left|\Lambda_n + 2\lambda\right|} + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{\left(i\Lambda_\alpha\right)^j\left|V_{n\alpha}\right|}{\left|\Lambda_n + 2\lambda\right|} = \\ &= \pm\left(i\lambda\right)^j + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{\left|V_{n\alpha}\right|}{\left|\Lambda_n + 2(Re\lambda + iIm\lambda)\right|} = \\ = 1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}&\frac{\left|V_{n\alpha}\right|}{\sqrt{\left(\Lambda_n + 2Re\lambda\right)^2 + 4Im^2\lambda}} \le 1 + \frac{1}{\left|Im\lambda\right|}\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{\Lambda_\alpha\left|V_{n\alpha}\right|}{\Lambda_n}\end{aligned}$$

Since

$$\left|f_2^{\pm(j)}(0,\lambda)\right| = \pm(\lambda)^j + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{(\lambda)^j\left|V_{n\alpha}\right|}{\left|\Lambda_n-2i\lambda\right|} + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{(i\Lambda_\alpha)^j\left|V_{n\alpha}\right|}{\left|\Lambda_n-2i\lambda\right|} =$$
$$= |\lambda|^j + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{|\lambda|^j\left|V_{n\alpha}\right|}{\sqrt{(\Lambda_n+2Im\lambda)^2+4Re^2\lambda}} + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{(|\Lambda_\alpha|)^j\left|V_{n\alpha}\right|}{\sqrt{(\Lambda_n+2Im\lambda)^2+4Re^2\lambda}} \le$$
$$\le |\lambda|^j + \frac{1}{|Re\lambda|}\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}|\lambda|^j\frac{\Lambda_\alpha\left|V_{n\alpha}\right|}{\Lambda_n} + \frac{1}{|Re\lambda|}\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{(\Lambda_\alpha)^2\left|V_{n\alpha}\right|}{\Lambda_n} \le$$
$$\le |\lambda|^j\,(1+\frac{1}{|Re\lambda|}\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{\Lambda_\alpha\left|V_{n\alpha}\right|}{\Lambda_n}) + \frac{1}{|Re\lambda|}\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{(\Lambda_\alpha)^2(\Lambda_\alpha-\Lambda_n)\left|V_{n\alpha}\right|}{\Lambda_n(\Lambda_\alpha-\Lambda_n)} \le$$
$$\le |\lambda|^j\,(1+\frac{1}{|Re\lambda|}\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{\Lambda_\alpha\left|V_{n\alpha}\right|}{\Lambda_n}) + \frac{1}{|Re\lambda|}\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{\Lambda_\alpha}{\Lambda_n(\Lambda_\alpha-\Lambda_n)}\Lambda_\alpha(\Lambda_\alpha-\Lambda_n)\left|V_{n\alpha}\right| \le$$
$$\le |\lambda|^j\,(1+\frac{1}{|Re\lambda|}\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{\Lambda_\alpha\left|V_{n\alpha}\right|}{\Lambda_n}) + \frac{2}{|Re\lambda|}\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\Lambda_\alpha(\Lambda_\alpha-\Lambda_n)\left|V_{n\alpha}\right|$$

Therefore, as $|\lambda|\to\infty$, we obtain

$$\begin{cases} f_1^{\pm(j)}(0,\lambda) = \pm(i\lambda)^j\,C_1 + o\,(1)\,, & for\,|\lambda|\to\infty,\;\; j=0,1,\;\; C_1>0 \\ f_2^{\pm(j)}(0,\lambda) = \pm(\lambda)^j\,C_2 + o\,(1)\,, & for\,|\lambda|\to\infty,\;\; j=0,1,\;\; C_2>0 \end{cases}$$

Then we get the coefficients defined by the formula (17) $A(\lambda)$,$B(\lambda)$,$C(\lambda)$,$D(\lambda)$ satisfies the following asymptotic:

$$A(\lambda) = \frac{1-i}{2} + o\,(1)\;\;\lambda\in IV\,quadrant$$
$$B(\lambda) = \frac{1+i}{2} + o\,(1)\;\;\lambda\in III\,quadrant$$
$$C(\lambda) = \frac{1+i}{2} + o\,(1)\;\;\lambda\in I\,quadrant$$
$$D(\lambda) = \frac{1-i}{2} + o\,(1)\;\;\lambda\in II\,quadrant. \tag{24}$$

From this, it follows that the zeros of the functions $A\,(\lambda)\,, B\,(\lambda)\,, C\,(\lambda)$ and $D\,(\lambda)$ from the sectors $S_k,\;k=0,1,2,3$ respectively are finite. □

Theorem 3. *The continuous spectra of the operator L consists of axis $Im\lambda=0$ and the continuous spectra may have spectral singularities at the points $\pm(\frac{\Lambda_n}{2})^2$.*

Proof. Since the resolvent exists, we can say that L is a closed operator. That's why we will only prove that the range of the operator $L-\lambda^2I$ is dense everywhere in $L_2\,(-\infty,+\infty)$ when $\lambda^2\in\{Im\lambda=0\}$ and doesn't coincide with the whole space, because the operator L doesn't have eigenvalues.

Taking into account Remark 2, we will investigate the function

$$R\,(x,t,\lambda) = \begin{cases} R_{11}\,(x,t,\lambda) & \lambda\in I\,quadrant \\ R_{12}(x,t,\lambda) & \lambda\in\;II\,quadrant \\ R_{21}\,(x,t,\lambda) & \lambda\in\;III\,quadrant \\ R_{22}\,(x,t,\lambda) & \lambda\in IV\,quadrant \end{cases}$$

in the neighborhood of poles λ^* from $\{Im\lambda=0\}$. Without loss of generality, we can investigate $R\,(x,t,\lambda)$ in the neighborhood of poles λ_0^* from $\{Im\lambda=0\}$. Then the number λ_0^* coincides with one of the numbers $\pm\frac{\Lambda_n}{2}$. From (20) – (23) it follows that $\lim\limits_{\lambda\to\lambda_0^*}(\lambda-\lambda_0^*)R(x,t,\lambda) = R_0(x,t)$ exists and $R_0(x,t)$ is a bounded function with respect to all the variables.

Let $\theta(x)$ be an arbitrary finite function. Then $\phi(x) = \int_{-\infty}^{\infty} R_0(x,t)\,\theta(t)dt$ is a bounded solution of Equation (7) for $\lambda = \frac{\Lambda_n}{2}$. Therefore $\phi(x) = C_0 f_1^+(x,\lambda_0^*)$. Comparison of the last relation with formulas (20) – (23) shows that if $\lambda_0^* \neq \frac{\Lambda_n}{2}$ then $C_0 = 0$ and so the kernel of the resolvent has removable singularity at the point λ_0^*. So there is a case $\lambda_0^* = \frac{\Lambda_n}{2}, n \in N$ where the kernel of resolvent has poles of the first order. Since $f_1^+(x,\lambda_0^*) \notin L_2(-\infty,\infty)$ then $\lambda_0^2 = \left(\frac{\Lambda_n}{2}\right)^2$ is a spectral singularity of the operator L according to (Naimark,1967). (Analogously we can show that $\lambda_0^2 = -\left(\frac{\Lambda_n}{2}\right)^2$ are spectral singularities of the operator L).

Suppose that there exists a function $\psi(x) \in L_2(-\infty,+\infty)$ such that $\psi(x) \neq 0$ and

$$\int_{-\infty}^{\infty} (Lf-\lambda f)\,\overline{\psi}(x)\,dx = 0 \tag{25}$$

for any $f(x) \in D(L)$.

From (25) we get that $\psi(x) \in D(L^*)$ and $\psi(x)$ are eigenfunctions of the operator L^* corresponding to the eigenvalue λ. More precisely, $\overline{\psi}(x)$ is the solution of the equation

$$z'' + q(x)\,z = \lambda z \tag{26}$$

in $L_2(-\infty,\infty)$. We get that $\psi(x) \equiv 0$ since the operator generated by the expression in the left part of (26) is a L type operator. We got a contradiction. Therefore, the range of the operator is dense everywhere on $L_2(-\infty,\infty)$.

Suppose that $\psi(x)$ is a finite function belonging to the range of the operator $\left(L-\lambda^2 I\right)$, $\lambda \neq 0,\ \lambda \neq \pm\frac{\Lambda_n}{2},\ \lambda \neq \pm\frac{i\Lambda_n}{2}\ , n \in N$, in other words there exist $y(x) \in D(L)$ such that

$$l(y) - \lambda y = \psi(x).$$

It is easy to see that $y^{(\tau)} \to 0$ when $|x| \to \infty,\ s = 0,1$.

Now let $\eta(x,\lambda)$ be the bounded solution of (26) when $\lambda \in\{Im\lambda = 0\}$. It is followed by theorem 1 that such a solution exists. Then the following equation is satisfied:

$$\int_{-\infty}^{\infty} (ly-\lambda y)\ \eta(x,\lambda)\,dx = \int_{-\infty}^{\infty} \psi(x)\ \eta(x,\lambda)\,dx =$$

$$= \int_{-\infty}^{\infty} y\left[\eta'' + q(x)\ \ \eta(x,\lambda) - \lambda\,\eta(x,\lambda)\right]\,dx = 0$$

This equation can be obtained by integrating by parts. From here we get that if

$$\psi(x) = \begin{cases} \overline{\eta(x,\lambda)} & , |x| \leq a \\ 0 & , |x| > a \end{cases}$$

where $a > 0$ then

$$\int_{-\infty}^{\infty} \psi(x)\ \eta(x,\lambda)\,dx = \int_{-a}^{a} |\eta(x,\lambda)|^2\ dx \neq 0,$$

in other words $\psi(x) \notin R\left(L-\lambda^2 I\right)$

Thus $R\left(L-\lambda^2 I\right)$ doesn't coincide with the whole space which we needed to prove. $\square$

Now taking $f_n^{\pm}(x) = V_{nn} f_1^{\mp}\left(x, \mp\frac{\Lambda_n}{2}\right)$ and (24) into account, we calculate

$$\lim_{\lambda \to \frac{\Lambda_n}{2}} (\Lambda_n - 2\lambda) R_{11}(x,t,\lambda) = \lim_{\lambda \to \frac{\Lambda_n}{2}} (\Lambda_n - 2\lambda) \tfrac{1}{2i\lambda} f_1^+(x,\lambda) \times$$

$$\times \left[f_1^+(t,\lambda) \tfrac{W\left[f_2^+, f_1^-\right]}{W\left[f_1^+, f_2^+\right]} + f_1^+(x,\lambda) f_1^-(t,\lambda) \right] = \tag{27}$$

$$= \tfrac{1}{i\Lambda_n} V_{nn} f_1^+\left(x, \tfrac{\Lambda_n}{2}\right) f_1^+\left(t, \tfrac{\Lambda_n}{2}\right) + \tfrac{1}{i\Lambda_n} V_{nn} f_1^+\left(x, \tfrac{\Lambda_n}{2}\right) f_1^+\left(t, \tfrac{\Lambda_n}{2}\right) =$$
$$= \tfrac{2}{i\Lambda_n} V_{nn} f_1^+\left(x, \tfrac{\Lambda_n}{2}\right) f_1^+\left(t, \tfrac{\Lambda_n}{2}\right)$$

Analogously we can calculate that

$$\lim_{\lambda \to \frac{i\Lambda_n}{2}} (\Lambda_n - 2i\lambda) R_{12}(x,t,\lambda) = \frac{2}{i\Lambda_n} V_{nn} f_2^-\left(x, \frac{i\Lambda_n}{2}\right) f_2^-\left(t, \frac{i\Lambda_n}{2}\right), \tag{28}$$

$$\lim_{\lambda \to -\frac{\Lambda_n}{2}} (\Lambda_n + 2\lambda) R_{21}(x,t,\lambda) = \frac{2}{i\Lambda_n} V_{nn} f_1^-\left(x, -\frac{\Lambda_n}{2}\right) f_1^-\left(t, -\frac{\Lambda_n}{2}\right) \tag{29}$$

$$\lim_{\lambda \to -\frac{i\Lambda_n}{2}} (\Lambda_n + 2i\lambda) R_{22}(x,t,\lambda) = \frac{2}{i\Lambda_n} V_{nn} f_2^+\left(x, -\frac{i\Lambda_n}{2}\right) f_2^+\left(t, -\frac{i\Lambda_n}{2}\right) \tag{30}$$

4 Eigenfunction expansions

Let L be the operator generated by $\frac{1}{\rho(x)}\left\{-\frac{d^2}{dx^2} + q(x)\right\}$ in the space $L_2(-\infty, +\infty, \rho(x))$

We proved that when $Im\lambda \geq 0,\ Re\lambda \geq 0$ the kernel of the resolvent of the operator L is in the form of

$$R_{11}(x,t,\lambda) = -\frac{1}{2i\lambda C(\lambda)} \begin{cases} f_1^+(x,\lambda) f_2^+(t,\lambda) & when\ t \leq x \\ f_1^+(t,\lambda) f_2^+(x,\lambda) & when\ t > x \end{cases} \tag{31}$$

Lemma 2. *Let $\psi(x)$ be an arbitrary twice differentiable continuous function belonging to $L_2(-\infty, \infty, \rho(x))$. Then*

$$\int_{-\infty}^{\infty} R(x,t,\lambda)\rho(t)\psi(t)\,dt = -\frac{\psi(x)}{\lambda^2} + \frac{1}{\lambda^2}\int_{-\infty}^{\infty} R(x,t,\lambda) g(t)\,dt \tag{32}$$

where

$$g(x) = -\psi''(x) + q(x)\psi(x) \in L_2(-\infty, \infty)$$

Proof. From (31) we get that

$$\int_{-\infty}^{+\infty} R_{11}(x,t,\lambda)\rho(t)\psi(t)\,dt = -\frac{f_1^+(x,\lambda)}{2i\lambda C(\lambda)} \int_{-\infty}^{x} \left\{-\frac{1}{\lambda^2} f_2^{+\prime\prime}(t,\lambda) + \frac{1}{\lambda^2} q(t) f_2^+(t,\lambda)\right\} \psi(t)\,dt -$$
$$- \frac{f_2^+(x,\lambda)}{2i\lambda C(\lambda)} \int_{x}^{\infty} \left\{-\frac{1}{\lambda^2} f_1^{+\prime\prime}(t,\lambda) + \frac{1}{\lambda^2} q(t) f_1^+(t,\lambda)\right\} \psi(t)\,dt$$

By integrating this identity twice by parts method and using (17) we obtain that

$$\int_{-\infty}^{\infty} R(x,t,\lambda)\rho(t)\psi(t)\,dt = -\frac{\psi(x)}{\lambda^2} + \frac{1}{\lambda^2}\int_{-\infty}^{\infty} R(x,t,\lambda) g(t)\,dt$$

where

$$g(x) = -\psi''(x) + q(x)\psi(x) \in L_2(-\infty, \infty)$$

It's easy to prove that if the conditions of the lemma satisfies, then when $|\lambda| \to \infty$

$$\int_{-\infty}^{\infty} R(x,t,\lambda)\rho(t)\psi(t)\,dt = -\frac{\psi(x)}{\lambda^2} + \overline{O}\left(\frac{1}{\lambda^2}\right) \tag{33}$$

Lemma is proved. $\square$

Integrating both hand sides of (32) along the circle $|\lambda| = R$ and passing to the limit as $R \to \infty$ we get

$$\psi(x) = -\lim_{R\to\infty} \frac{1}{2\pi i} \int_{|\lambda|=R} 2\lambda d\lambda \int_{-\infty}^{\infty} R(x,t,\lambda)\,\rho(t)\,\psi(t)\,dt$$

The function $\int_{-\infty}^{\infty} R(x,t,\lambda)\,\rho(t)\,\psi(t)\,dt$ is analytical inside the contour with respect to λ excepting the points $\lambda = \lambda_n,\ n \in N,\ \lambda = \pm\frac{\Lambda_n}{2},\ \lambda = \pm\frac{i\Lambda_n}{2}$.

Let $\Gamma_0^+\left(\Gamma_0^-\right)$ denote the contour formed by the segments $\left[0, \frac{\Lambda_1}{2} - \delta\right], ..., \left[\frac{\Lambda_n}{2} - \delta, \frac{\Lambda_n}{2} + \delta\right]$ $\left(\left[0, -\frac{\Lambda_1}{2} - \delta\right], ..., \left[-\frac{\Lambda_n}{2} - \delta, -\frac{\Lambda_n}{2} + \delta\right]\right)$ and the semicircles of the radius δ with centers at the points $\frac{\Lambda_n}{2}\left(-\frac{\Lambda_n}{2}\right)$ located in the upper (lower) half-plane and $\Gamma_i^+\left(\Gamma_i^-\right)$ $i = 1,2,3$ denote the contour formed by rotating $\Gamma_i^+\left(\Gamma_i^-\right)$ by angle $\frac{i\pi}{2}$ $i = 1,2,3$. Then

$$\begin{aligned}
\psi(x) = &-\frac{1}{2i\pi}\int_{\Gamma_0^-} 2\lambda d\lambda \int_{-\infty}^{+\infty} \left[R_{22}(x,t,\lambda) - R_{11}(x,t,\lambda)\right]\rho(t)\,\psi(t)\,dt- \\
&-\frac{1}{2i\pi}\int_{\Gamma_1^-} 2\lambda d\lambda \int_{-\infty}^{+\infty} \left[R_{11}(x,t,\lambda) - R_{12}(x,t,\lambda)\right]\rho(t)\,\psi(t)\,dt- \\
&-\frac{1}{2i\pi}\int_{\Gamma_2^-} 2\lambda d\lambda \int_{-\infty}^{+\infty} \left[R_{12}(x,t,\lambda) - R_{21}(x,t,\lambda)\right]\rho(t)\,\psi(t)\,dt- \\
&-\frac{1}{2i\pi}\int_{\Gamma_3^-} 2\lambda d\lambda \int_{-\infty}^{+\infty} \left[R_{21}(x,t,\lambda) - R_{22}(x,t,\lambda)\right]\rho(t)\,\psi(t)\,dt+ \\
&+\operatorname*{Res}_{\lambda=-\frac{i\Lambda_n}{2}} R_{22}(x,t,\lambda) + \operatorname*{Res}_{\lambda=\frac{\Lambda_n}{2}} R_{11}(x,t,\lambda) + \operatorname*{Res}_{\lambda=\frac{i\Lambda_n}{2}} R_{12}(x,t,\lambda) + \operatorname*{Res}_{\lambda=-\frac{\Lambda_n}{2}} R_{21}(x,t,\lambda) + \\
&+\sum_{n=1}^{l} Res\left(2\lambda \int_{-\infty}^{+\infty} R(x,t,\lambda)\,\rho(t)\,\psi(t)\right)\Bigg|_{\lambda=\alpha_n}
\end{aligned}$$

Let's calculate every term separately:

$$R_{11}(x,t,\lambda) - R_{22}(x,t,\lambda) = \frac{f_2^+(x,\lambda)\,f_2^+(t,\lambda)}{2i\lambda A(\lambda)\,C(\lambda)},$$

$$R_{21}(x,t,\lambda) - R_{12}(x,t,\lambda) = \frac{f_2^-(x,\lambda)\,f_2^-(t,\lambda)}{2i\lambda B(\lambda)\,D(\lambda)}$$

$$R_{12}(x,t,\lambda) - R_{11}(x,t,\lambda) = \frac{f_1^+(x,\lambda)\,f_1^+(t,\lambda)}{2\lambda C(\lambda)\,D(\lambda)}$$

$$R_{22}(x,t,\lambda) - R_{21}(x,t,\lambda) = \frac{f_1^-(x,\lambda)\,f_1^-(t,\lambda)}{2\lambda A(\lambda)\,B(\lambda)}.$$

The residues of the resolvents $R_{11}(x,t,\lambda)$, $R_{12}(x,t,\lambda)$,$R_{21}(x,t,\lambda)$ and $R_{22}(x,t,\lambda)$ at $\lambda = \frac{\Lambda_n}{2}$, $\lambda = \frac{i\Lambda_n}{2}$, $\lambda = -\frac{\Lambda_n}{2}$, $\lambda = -\frac{i\Lambda_n}{2}$ respectively are equal to the limits which we calculated in (27) – (30). Then for any function $\psi(x) \in L_2(-\infty,\infty,\rho(x))$ we have the following eigenfunction expansion:

$$\psi(x) = -\frac{1}{2\pi}\int_{\Gamma_0^+}\frac{1}{A(\lambda)C(\lambda)}d\lambda\int_{-\infty}^{+\infty} f_2^+(x,\lambda) f_2^+(t,\lambda)\rho(t)\psi(t)\,dt$$

$$+\frac{1}{2\pi}\int_{\Gamma_2^+}\frac{1}{B(\lambda)D(\lambda)}d\lambda\int_{-\infty}^{+\infty} f_2^-(x,\lambda) f_2^-(t,\lambda)\rho(t)\psi(t)\,dt$$

$$+\frac{1}{2i\pi}\int_{\Gamma_1^+}\frac{1}{C(\lambda)D(\lambda)}\int_{-\infty}^{+\infty} f_1^+(x,\lambda) f_1^+(t,\lambda)\rho(t)\psi(t)\,dt$$

$$-\frac{1}{2i\pi}\int_{\Gamma_3^+}\frac{1}{A(\lambda)B(\lambda)}d\lambda\int_{-\infty}^{+\infty} f_1^-(x,\lambda) f_1^-(t,\lambda)\rho(t)\psi(t)\,dt \tag{34}$$

$$+\frac{2}{i\Lambda_n}V_{nn}f_1^+\left(x,\frac{\Lambda_n}{2}\right)f_1^+\left(t,\frac{\Lambda_n}{2}\right)+\frac{2}{i\Lambda_n}V_{nn}f_2^-\left(x,\frac{i\Lambda_n}{2}\right)f_2^-\left(t,\frac{i\Lambda_n}{2}\right)$$

$$+\frac{2}{i\Lambda_n}V_{nn}f_1^-\left(x,-\frac{\Lambda_n}{2}\right)f_1^-\left(t,-\frac{\Lambda_n}{2}\right)+\frac{2}{i\Lambda_n}V_{nn}f_2^+\left(x,-\frac{i\Lambda_n}{2}\right)f_2^+\left(t,-\frac{i\Lambda_n}{2}\right)$$

$$+\textstyle\sum_{n=1}^{l} Res\left(2\lambda\int_{-\infty}^{+\infty}R(x,t,\lambda)\rho(t)\psi(t)\right)\Big|_{\lambda=\Lambda_n}$$

5 Inverse problem

From (27) – (30) it follows that the kernels of the resolvent $R_{11}(x,t,\lambda)$, $R_{22}(x,t,\lambda)$, $R_{21}(x,t,\lambda)$ and $R_{12}(x,t,\lambda)$ admits meromorphic continuations in clockwise direction to the next quadrant and may have poles at the points $\pm\frac{\Lambda_n}{2}$, $\pm\frac{i\Lambda_n}{2}, n \in N$ outside of S_k. Here

$$S_k = \left\{\frac{k\pi}{2} < \arg\lambda < \frac{(k+1)\pi}{2}\right\},\ k = 0,1,2,3.$$

The poles of the resolvent are called quasi – stationary states of the operator L. In spectral expansion (34) the numbers V_{nn}, $n \in N$, play as a part of normalizing corresponding to quasieigenfunctions of the operator L. So we can formulate the inverse problem about the reconstruction of the potential of the equation (7).

From Theorem 1 it is easy to see that the functions $f_1^{\pm}(x,\lambda)$ are not defined at the points $\mp\frac{\Lambda_n}{2}$.

The following limit

$$\overset{\approx}{\lim_{\lambda\to\mp\frac{\Lambda_n}{2}}}\left(\lambda\pm\frac{\Lambda_n}{2}\right)f_1^{\pm}(x,\lambda) = \begin{cases} 0, & \lambda\notin M \\ f_n^{\pm}(x), & \lambda=\lambda_n,\ n\in N \end{cases} \tag{35}$$

exists and is uniform in X. Here the symbol $\overset{\approx}{\lim}$ means the limit is non - tangent direction when λ tends to $\lambda^* = \{\mp\frac{\Lambda_n}{2}, \mp\frac{i\Lambda_n}{2}\}$ in such a way that $+\delta < \arg(\lambda-\lambda^*) < \pi - \delta$ for arbitrary but fixed $\delta > 0$. The function $f_n^{\pm}(x)$ is a solution of (7) with $\lambda = \mp\frac{\Lambda_n}{2}$ linearly dependent on $f_1^{\mp}\left(x,\mp\frac{\Lambda_n}{2}\right)$. Consequently, there exists a complex s_n, $n \in N$ such that

$$f_n^{\pm}(x) \equiv s_n f_1^{\mp}\left(x,\mp\frac{\Lambda_n}{2}\right) \tag{36}$$

As in Simbirskii (1992) we can the following definition.

Definition 1. *The set $\{s_n\}_{n\in N}$ is called the spectral data set of the operator L.*

It follows from (36) that

$$e^{-i\frac{\Lambda_n}{2}x}\sum_{\alpha:\alpha>>n}V_{n\alpha}e^{i\Lambda_\alpha x}\equiv s_n e^{i\frac{\Lambda_n}{2}x}\left(1+\sum_{m\in N}\frac{2}{\Lambda_n+\Lambda_m}\sum_{\alpha:\alpha>>m}V_{m\alpha}e^{i\Lambda_\alpha x}\right) \tag{37}$$

and by virtue of uniqueness theorem for almost – periodic functions, we have

$$\begin{array}{ll} V_{n,n}=-s_n, & n\in N \\ V_{n,\alpha}=-s_n\sum_{m:m<<\alpha\oplus n}\frac{2V_{m,\alpha\oplus n}}{\Lambda_n+\Lambda_m}, & \alpha>n \end{array}$$

Let a set of numbers $\{s_n\}_{n\in N}$ be given. We shall construct a potential $q(x)\in Q^p(M)$ such that the spectral data set of the operator L with this potential is identical with the set $\{s_n\}_{n\in N}$. In (36) comparing the formulas for these functions, we see that $s_n=V_{nn}$, therefore

$$f_n^{\pm}(x)=V_{nn}f_1^{\mp}\left(x,\mp\frac{\Lambda_n}{2}\right) \tag{38}$$

Inverse problem: Given the spectral data $\{s_n\}_{n\in N}$ construct the potential $q(x)\in Q^p(M)$. Using the results obtained above, we arrive at the following procedure for the solution of the inverse problem:

1. Taking into account (38), calculate

$$V_{n\alpha+n}=V_{nn}\sum_{m=1}^{\alpha}\frac{V_{m\alpha}}{m+n}$$

   from which all the numbers $V_{n\alpha},\ \alpha=1,2,...;\ n=1,2,......;\ n<\alpha$ are defined.

2. From recurrent formula (10), (11) find all numbers q_n.

So, the inverse problem has a unique solution and the numbers $V_{n\alpha},\ \alpha=1,2,...;\ n=1,2,......;\ n<\alpha$ and q_n are defined constructively by spectral data.

Thus we arrive at the following theorem.

Theorem 4. *The specification of spectral data uniquely determines the potential $q(x)$.*

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